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Assignment Messages and Exchanges†

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"Assignment messages" are maximally general messages to describe substitutable preferences by means of a linear program. With "integer assignment messages," there exist integer-valued Walrasian allocations, extending a result of Lloyd S. Shapley and Martin Shubik (1971). Any pure Nash equilibrium profile of the Walrasian mechanism with participants limited to assignment messages is also a Nash equilibrium of the unrestricted Walrasian mechanism. Assignment exchanges are generalizations of single-product double auctions and are related to ascending multi-product clock auctions and the Vickrey mechanism. Assignment messages also have additional applications in mechanism design. (*JEL* D44, D82)

In abstract mechanism theory, the designer is often presumed able to create a direct mechanism in which each participant reports its "type," revealing the participant's preferences along with anything else the participant may know. In practice, these details can be too numerous to report. For example, in Federal Communciations Commission (FCC) Auction No. 66 with 1,132 licenses for sale, a type includes a vector of values for every subset of licenses. Reporting that vector would have entailed reporting 2^{1132} numbers.

One approach to mitigating the length-of-report problem is to simplify reporting by limiting the message space. The National Resident Matching Program uses this approach. It limits hospitals' reports to a number of positions and a rank order list of candidates. If a hospital has 10 openings and interviews 50 candidates, it reports the number 10 and a list of 50—a manageably short message. In contrast, because the number of classes of 10 or fewer doctors from among 50 is about 1.3×10^{10}, a general type report, including a rank order list of all those classes, would be impracticably long.

This paper introduces and analyzes a new message space—the space of *assignment messages*—designed for use in auctions, exchanges, and other applications where goods are substitutes. Assignment messages describe preferences indirectly as the value of a linear program for which the set of constraints is describable as a

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structured collection of trees or hierarchies. We show that if the constraints have this form, then the goods are substitutes, regardless of the various parameters. Conversely, if the constraints describing substitution among different goods do not respect the tree structure, then there exist parameters such that goods are not substitutes. In that sense, the constraint structure employed by assignment messages is the most general one consistent with substitutable preferences in linear programming.

An *assignment exchange* is a simplified direct Walrasian mechanism in which participants are restricted to report their preferences using assignment messages. The properties of assignment exchanges are discussed below.

Among the parameters reported by a bidder in an assignment message are ones that specify local rates of technical substitution among goods.[1] *Integer assignment messages* restrict those rates to be zero or one, and restrict any bounds on groups of quantities to be integers. If all traders' preferences can be described in this way, then there is an efficient allocation that is an integer vector.[2] Consequently, the *integer assignment exchange*, which is the assignment exchange restricted to integer assignment messages, transacts in integer quantities.

The assignment exchange shares important aspects of its price and payoff structure with its namesake, the assignment mechanism of Lloyd S. Shapley and Martin Shubik (1971).[3] The integer assignment exchange has the further property that all equilibrium quantities are integers and extends the Shapley-Shubik mechanism in three important ways. First, participants in an integer assignment exchange may buy or sell multiple types of goods simultaneously, instead of just one type. Second, they may trade any integer number of units of each type of good, instead of just one unit. And third, they may buy some goods and sell others, instead of being restricted to just one role as a buyer or a seller.

The integer allocation property can be important for a variety of applications, including those in which commodities are shipped most efficiently by the truckload or container. Even when goods are perfectly divisible, contracts are often denominated and traded in whole numbers of units, so the ability to respect integer constraints may be useful even in those applications.

The restriction of local rates of technical substitution to zero or one is a strong one, but it is surprisingly often a reasonable approximation for practical applications. For example, an electric utility delivering retail power to its customers might acquire wholesale power from generators at three different locations, A, B, and C, but may be limited in its ability to utilize power from each source by its source-specific transmission capacities. When additional transmission capacity is available at source A, one unit of power from A can substitute for one unit from any other source. When

[1] Strictly speaking, because the model is one of preferences rather than production, rates of "technical substitution" are not defined. However, assignment messages report constraints resembling production constraints as well as parameters to determine the slopes of those constraints, so it is convenient and intuitive to describe the slopes of constraints using the language of producer theory.

[2] When bundles necessarily consist of integer quantities and goods are substitutes, a version of the limited one-for-one substitution property is implied. See Faruk Gul and Ennio Stacchetti (1999) and Milgrom and Bruno Strulovici (2009).

[3] In both mechanisms, goods are substitutes, and the set of market-clearing goods prices is a nonempty, closed, convex sublattice. Consequently, there is a seller-best, buyer-worse equilibrium price vector and a seller-worst, buyer-best equilibrium price vector.

capacity is not available, an additional unit of power at A is unusable. It replaces zero units of power from other sources. Similarly, a cereal maker may be able to substitute bushels of grain delivered today for bushels delivered tomorrow up to a limit imposed by its grain-storage capacity, or it may substitute one unit of a particular type or grade of grain for one of another type within limits specified by the product-formulation requirements. A similar substitution pattern is sometimes found among sellers, as when a manufacturer can deliver several versions of the same processed good in a total amount that is limited by the overall capacity of its factory.[4] This pattern of *limited one-for-one substitution* can be a useful approximation whenever lots differ in attributes such as time and location of availability, grade, degree of processing, delivery and contract terms, or some combination of these.

General assignment messages extend the integer assignment messages by allowing participants to specify local rates of technical substitution besides zero and one. For example, in markets for electric power, if the transmission losses in shipping power from A are higher than from B, then one unit of power from A replaces less than one unit from B—the rate of technical substitution is positive but less than one. Using integer assignment messages, a bidder can account for such transmission losses only approximately, by treating the power from different sources as having different money values, but general assignment messages allow an exact representation.

An important attribute of assignment messages is that they allow not only bids to buy or sell one of several different goods, but also "swap" bids. For example, in a securities market, a swap could specify that an offer to buy shares of stock is executed only if an offer to sell certain call options on that stock is also executed. Such a linkage can be especially valuable in markets with limited liquidity because it eliminates execution risk.[5]

The ability to report swap bids makes the integer assignment exchange applicable to some resource allocation problems involving *complementary* goods for which package exchange mechanisms might have been thought to be necessary.[6] This is, perhaps, surprising given that assignment messages can directly only express substitutable preferences. Figure 1 displays an example.

Points A, B, and C, in Figure 1, represent physical locations (in southeast Wyoming) where wind farms produce electrical power carried by new long-range transmission lines. Point D represents a node (in northwest Colorado) where the power is injected into the existing transmission grid. For a producer located at A, transmission capacity along lines AC and CD are Leontief complements; the producer is constrained by the minimum of the capacity acquired on AC or CD. Similarly, producers at B regard BC and CD as Leontief complements. The power producers located at A, B, and C compete to acquire capacity on the CD link. Let us assume that there are one

[4] The National Resident Matching Program, with its fixed number of slots at each hospital, imposes one-for-one substitution but excludes resident wages from the process. An assignment auction could be suitable for that application, provided that wages are made endogenous. Vincent P. Crawford (2008) proposes a simultaneous ascending auction mechanism for the same application.

[5] Some traders call this "leg risk" because the danger is that one "leg" of a transaction is executed while the other is not.

[6] See Milgrom (2007) for an introduction to the economic package allocation problem; Noam Nisan (2006) for an analysis of some message spaces that might be used in package auctions; and Peter Cramton, Yoav Shoham, and Richard Steinberg (2006) for a collection of related articles.

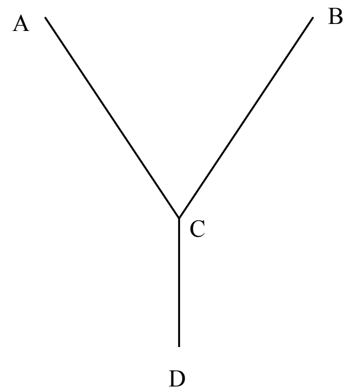

Figure 1. A Y-Shaped Electrical Transmission Grid

or more separate capacity suppliers for each link and that the costs for any suppliers that can supply more than one link are additively separable across links.

Despite the technical complementarities among successive links, preferences of both buyers of transmission links and suppliers of capacity can be expressed using integer assignment messages. The key lies in the way *lots* are defined. Suppose the exchange is organized to trade three kinds of lots. Each lot is a package of links sufficient to transmit a unit of energy from one of the points A, B, or C to point D (AD, BD, or CD, respectively). With lots defined in that way, each energy producer/capacity buyer can bid on the lot connecting its location to the root at D, so these participants can express their preferences accurately. A supplier who wishes to offer capacity on one of the single links AC or BC can do that using a *swap* bid that links offers to buy and sell. For example, an offer to sell capacity on AC at a price of at least X is represented as a swap that links an offer to sell capacity on the AD lot with a bid to buy equal capacity on the CD lot at a price difference of at least X. Thus, with the specified lots, both buyers and sellers can express preferences accurately. The theorems about assignment exchanges apply. Despite complementarities and indivisible lots, which often preclude the existence of supporting prices, this is a special case in which the existence of market-clearing prices is guaranteed.[7]

Restricting the messages available to participants in a mechanism can affect incentives and performance. In a general simplification, some message profiles may be equilibria of the simplified mechanism even though they were not equilibria of the original, extended mechanism. A *tight* simplification is one with the property that, for every profile of participant preferences in some specified set and every $\varepsilon \geq 0$, all of the full-information, pure ε-Nash equilibria of the simplified mechanism are also ε-Nash equilibria of the original mechanism (see Milgrom (forthcoming)). Assignment exchanges are tight simplifications of general Walrasian exchange mechanisms for any preference that can be represented by a continuous, real-valued

[7] A similar construction can be used in any acyclic network by identifying one node in each component of the graph as a root, and expressing all lots in terms of flows from a node to a root. Demand need not be located only at the roots for this construction to work, but the demanded packages of links must lie in sequence on one side of the root.

utility function whose arguments are the bidder's assigned quantity vector and the price vector. Thus, even though participants may have preferences that are not well described by assignment messages, the restriction to assignment messages never introduces any pure ε-Nash equilibrium that was not already present in the full Walrasian mechanism.

The remainder of this paper is organized as follows. Section I introduces the assignment message space and reports three theorems about it. The first is that the assignment messages express only substitutable preferences. The second is that when all preferences are expressed by assignment messages, the set of market-clearing prices is a nonempty, closed, convex sublattice. The third is that if all participants' preferences are expressed with *integer assignment messages*, then there is an efficient allocation using only integer quantities of all goods. Section II provides a partial converse to two of these theorems. Assignment messages require that the constraints connecting different goods form a "tree." If that constraint is relaxed at all, then the conclusions of the first two theorems of Section I are no longer valid. Section III discusses tightness. Its main conclusion is that the assignment exchanges, as well as many further simplifications of these exchanges, are tight simplifications of a Walrasian mechanism. Section IV discusses the connections between the assignment exchange and two familiar mechanisms: the single-product double auction and the Vickrey auction. Section V discusses some of the most likely applications.

I. Assignment Messages

Consider a resource allocation problem with goods indexed by $k = 1,\ldots,K$ and participants are indexed by $n = 1,\ldots,N$. If participants' preferences are quasi-linear, then the utility for a trade is expressed as the value $V_n(q_n)$ of the bundle $q_n \in \Re^K$ acquired plus any net cash transfer. The set of demanded bundles at price vector p is $\arg\max_{q_n} V_n(q_n) - p \cdot q_n$, where q_n may include both positive and negative components. A direct mechanism must specify a message space for describing V_n. Assignment messages model demand as originating from multiple sources, describing each q_n as the sum of scalars x_j for $j \in J(n)$, where j is the serial number of a bid and $J(n)$ is the set of serial numbers for bids submitted by bidder n.

Formally, an assignment message consists of a collection of bids and constraints.[8] Each bid by bidder n consists of a 5-tuple $(k_j, v_j, \rho_j, l_j, u_j)$ where k_j identifies the type of product, v_j identifies the "value" of the bid, $\rho_j > 0$ identifies the "effectiveness," and the remaining two terms are lower and upper bounds on quantity: $l_j \leq 0 \leq u_j$. The role of the effectiveness coefficient, which is to allow general local rates of technical substitution, will be formalized shortly.

In addition to the bids, participant n's assignment message expresses quantity constraints of two kinds. First are the *single-product bid group constraints* for each good k:

$$l_{kS} \leq \sum_{j \in S} x_j \leq u_{kS} \quad \text{for } S \in \Im_{nk}, \tag{1}$$

[8] A related precursor to this message space is the space of *endowed assignment messages*, introduced by John William Hatfield and Milgrom (2005).

where $\Im_{nk}$ includes all singletons $S = \{j\}$ for which $k_j = k$ and may include other subsets of $R_{nk} = \{j \in J(n) \mid k_j = k\}$. For the singletons, $l_{kj\{j\}} \equiv l_j$ and $u_{kj\{j\}} \equiv u_j$. Second are the *multi-product bid group constraints* indexed by the set $\Im_{n0}$. These are of the form

$$l_{0S} \leq \sum_{j \in S} \rho_j x_j \leq u_{0S} \quad \text{for } S \in \Im_{n0}. \tag{2}$$

Unlike the sets used in the single-product group constraints, the sets $S \in \Im_{n0}$ may include bids on multiple products. Also, unlike the sums in (1), those in (2) are weighted by the effectiveness coefficients, to parameterize the rates of technical substitution among the different products. Note that these constraints can apply to bids to buy ($l_{kS} = 0$), bids to sell ($u_{kS} = 0$), bids to buy or sell ($l_{kS} < 0$, $u_{kS} > 0$), and swaps between multiple products ($l_{0S} = u_{0S} = 0$).

To simplify notation, we suppress the bidder index n while we are analyzing the reports and preferences of a single bidder. The index will reappear later when we analyze allocations for multiple participants. Using the bids and constraints, bidder n's message is interpreted to report a value for any feasible bundle of products $q = (q_1, \ldots, q_K)$ as follows:

$$\begin{aligned} V(q) = \max_x \sum_{j \in J} v_j x_j \quad & \text{subject to} \\ l_{kS} \leq \sum_{j \in S} x_j \leq u_{kS} \quad & \text{for } S \in \Im_k,\ k = 1, \ldots, K \\ l_{0S} \leq \sum_{j \in S} \rho_j x_j \leq u_{0S} \quad & \text{for } S \in \Im_0 \\ \sum_{j \in R_k} x_j = q_k \quad & \text{for } k = 1, \ldots, K. \end{aligned} \tag{3}$$

Because the vector $(q, x) \equiv 0$ satisfies all the constraints in (3), the zero bundle $q = 0$ is feasible. By a theorem of linear programming, the set of vectors q for which the problem is feasible is a closed, bounded, convex set $Q \subseteq \Re^K$, and V is a continuous, concave function on that set.

The next step is to put more structure on the single- and multi-product bid constraints to complete the definition of assignment messages. To describe this structure, we need to define three more concepts: trees, constraint forests, and extended predecessor functions.

As we described above, assignment messages allow two kinds of constraints. There is a set of constraints that describes substitution among products. These are required to form a tree. In addition, for each product k, there may be a set of constraints limiting the quantities assigned to each bid. These, too, must form a tree. Together, these trees form a constraint forest. To describe the relevant trees in a compact notation, we define an extended predecessor function that not only maps sets into their predecessors in the tree, but also maps bids into the smallest set in the tree that contains that bid. These concepts, and others essential to the theorems of this section, are defined below.

DEFINITIONS:

1. The *demand correspondence* for V is $D(p) \equiv \arg\max_{q \in Q} V(q) - p \cdot q$.

2. The *indirect profit function* for V is $\pi(p) \equiv \max_{q \in Q} V(q) - p \cdot q$.

3. The valuation V is *substitutable* if for all prices $p, p' \in \Re_{+}^{K}$ and all $k = 1, \ldots, K$, if $D(p) = \{x\}$ and $D(p_{-k}, p'_k) = \{x'\}$ are singletons, and $p'_k > p_k$, then $x'_{-k} \geq x_{-k}$.

4. A collection of sets $\Im$ is a *tree* if (1) for any two nondisjoint sets $S, S' \in \Im$, either $S \subset S'$ or $S' \subset S$ and (2) $\Im$ contains a largest set—the union of all its elements. That largest set is the *root* of $\Im$.

5. Given a tree of sets $\Im$, its *extended predecessor function* (P) maps each element of $\Im$, excluding the root R, into its unique predecessor (the smallest set in $\Im$ which contains it), and maps each $j \in R$ into the smallest set S satisfying $j \in S \in \Im$. Below, P_k denotes the extended predecessor function for tree $\Im_k$.

6. A *constraint forest* is a collection of trees and associated bounds $\{\Im_0, \ldots, \Im_K, \{(l_{kS}, u_{kS}) \mid S \in \Im_k, k = 0, \ldots, K\}\}$ with all $l_{kS} \leq 0 \leq u_{kS}$. The trees satisfy:

   (a) The root of $\Im_0$ is $R_0 = J$ and, for $k = 1, \ldots, K$, the root of $\Im_k$ is $R_k = \{j \in J \mid k_j = k\}$.

   (b) For $k = 1, \ldots, K$, the terminal nodes of tree $\Im_k$ are the singleton sets $\{j\}$ with $j \in J$ and $k_j = k$.

   (c) All bounds except the root bounds are finite, $0 \geq l_{kS} > -\infty$ and $0 \leq u_{kS} < +\infty$, but the bounds on the roots may be infinite, $0 \geq l_{kR_k} \geq -\infty$ and $0 \leq u_{kR_k} \leq +\infty$.

   (d) For any singleton set $\{j\} \in \Im_{k_j}$, $l_{k_j\{j\}} = l_j$ and $u_{k_j\{j\}} = u_j$.

7. An *assignment message* consists of a collection of bids $(k_j, v_j, \rho_j, l_j, u_j)$ and a constraint forest $\{\Im_0, \ldots, \Im_k, \{(l_{kS}, u_{kS}) \mid S \in \Im_k, k = 0, \ldots, K\}\}$.

8. An *integer assignment message* is an assignment message with each $\rho_j = 1$ and with all bounds l_{kS} and u_{kS} integers.

9. An *assignment exchange* is a mechanism mapping profiles of assignment messages for each bidder n to an outcome pair $(q_1^*, \ldots, q_N^*, p^*)$, where $q^* \in \arg\max_{\{q \mid q_n \in Q_n\}} \sum_{n=1}^{N} V_n(q_n)$ subject to $\sum_{n=1}^{N} q_{nk} = 0$ for $k = 1, \ldots, K$ and p^* is a supporting price

vector. That is, for $n = 1,\ldots,N$, $q_n^* \in \arg\max_{q\in Q_n}(V_n(q) - p^*\cdot q)$ (equivalently, $p^* \in \arg\min_p \pi_n(p) + p\cdot q_n^*$).

10. An *integer assignment exchange* is an assignment exchange in which the messages are restricted to be integer assignment messages.

The integer assignment messages extend the set of messages allowed by the Shapley-Shubik mechanism. In the Shapley-Shubik mechanism, each participant occupies just one role, as a buyer or a seller. Each seller message includes just one bid ($|J(n)| = 1$), and each buyer message includes just one bid for each product. If participant n is a seller, then the constraints on its one bid are $l_{n1} = -1$ and $u_{n1} = 0$. If participant n is a buyer, then its constraint bounds for each bid are $l_{nj} = 0$ and $u_{nj} = 1$, and its one multi-product group constraint has bounds $l_{n0R_{n0}} = 0$ and $u_{n0R_{n0}} = 1$. The integer assignment message space extends this Shapley-Shubik message space by allowing more bids, more constraints, and general integer bounds.

The three main results of this section can now be stated. Proofs follow.

THEOREM 1: *If participant n reports an assignment message, then its valuation* $V:q \to \Re$, *as given by* (3), *is continuous, concave, and substitutable, and its indirect profit function is submodular.*

THEOREM 2: *If every participant n reports a continuous, concave substitutable valuation on a convex, compact set* Q_n, *then the set of market-clearing prices for the report profile is* $\arg\min_p \sum_{n=1}^N \pi_n(p)$. *This set is a nonempty, closed, convex sublattice.*

THEOREM 3: *If every participant reports an integer assignment message, then there is an integer vector* $q^* \in \arg\max_{\{q\,|\,q_n\in Q_n\}} \sum_{n=1}^N V_n(q_n)$ subject to $\sum_{n=1}^N q_{nk} = 0$ for all k.

The proof of Theorem 1 makes use of two lemmas, which are of independent interest.

LEMMA 1: *Suppose that the valuation function V is such that the corresponding indirect profit function* π *is well defined. Then V is substitutable if and only if its indirect profit function* π *is submodular.*[9]

LEMMA 2: *Suppose* $\pi(p) = \min_z g(z)$ *subject to* $(z, p) \in S$, *where g is submodular, S is a sublattice in the product order, and p is a parameter. Then,* π *is submodular.*

[9] Earlier versions of this result, as in Lawrence M. Ausubel and Milgrom (2002) or Milgrom and Strulovici (2009), impose additional restrictions, such as discreteness of the goods, which are appropriate for those contexts. This version drops the unnecessary additional assumptions.

PROOF OF LEMMA 1:

Since π is convex on $\Re^K$, it is locally Lipschitz and differentiable almost everywhere. By Hotelling's lemma, the demand set is a singleton $D(p) = \{x(p)\}$ at exactly those points of differentiability, and $\pi_k(p) \equiv \partial\pi(p)/\partial p_k = -x_k(p)$. Substitutability is equivalent to the condition that for $k = 1,\ldots,K$, $x_k(p)$ is nondecreasing in $p_{k'}$ for $k' \neq k$. Submodularity is equivalent to the condition that, on the same domain, $\pi_k(p)$ is nonincreasing in $p_{k'}$ for $k' \neq k$.

PROOF OF LEMMA 2:

Let p and p' be two price vectors, and let z and z' be corresponding optimal solutions, so that $\pi(p) = g(z)$, $\pi(p') = g(z')$, and (z,p), $(z',p') \in S$. Since S is a sublattice, $(z\wedge z', p\wedge p'), (z\vee z', p\vee p') \in S$. By the definition of π, $\pi(p\wedge p') \leq g(z\wedge z')$, and $\pi(p\vee p') \leq g(z\vee z')$. Since g is submodular, $g(z\wedge z') + g(z\vee z') \leq g(z) + g(z')$. Hence, $\pi(p\wedge p') + \pi(p\vee p') \leq \pi(p) + \pi(p')$.

PROOF OF THEOREM 1:

We will use the dual program corresponding to (3) to show that the indirect profit function π satisfies the assumptions of Lemma 2.

In program (3), let λ^u_{kS} denote the dual price of the upper-bound (k,S)-constraint, λ^l_{kS} the dual price of the corresponding lower-bound constraint, and μ_k the dual price of the product k constraint. Since only one of λ^u_{kS} and λ^l_{kS} can be nonzero, both can be inferred from $\lambda_{kS} = \lambda^u_{kS} - \lambda^l_{kS}$. Using the duality theorem of linear programming (e.g., see David Gale (1960)) in the third inequality below, the indirect profit function corresponding to V is

$$
\begin{aligned}
(4)\quad \pi(p) &= \max_q V(q) - p\cdot q \\
&= \max_{q,x} \sum_{j\in J} v_j x_j - \sum_{k=1}^{K} p_k q_k && \text{subject to} \\
&\quad l_{kS} \leq \sum_{j\in S} x_j \leq u_{kS} && \text{for } S\in\Im_k,\ k = 1,\ldots,K \\
&\quad l_{0S} \leq \sum_{j\in S} \rho_j x_j \leq u_{0S} && \text{for } S\in\Im_0 \\
&\quad \sum_{j\in R_k} x_j - q_k = 0 && \text{for } k = 1,\ldots,K \\
&= \min_{\lambda,\mu} \sum_{k=0}^{K} \sum_{S\in\Im_k} (u_{kS}\,\lambda^u_{kS} - l_{kS}\,\lambda^l_{kS}) && \text{subject to} \\
&\quad \sum_{\{S\in\Im_k | j\in S\}} \lambda_{kS} + \mu_k + \rho_j\Bigg(\sum_{\{S\in\Im_0 | j\in S\}} \lambda_{0S}\Bigg) \geq v_j && \text{for all } j\in J,\ k = k_j \\
&\quad \mu_k \leq p_k && \text{for } k = 1,\ldots,K.
\end{aligned}
$$

A change of variables reveals the lattice structure in (4). For $k = 0$ and $S \in \Im_0$, define $\Lambda_{0S} = -\sum_{\{\hat{S}\in\Im_0 | S\subseteq\hat{S}\}}\lambda_{0\hat{S}}$ and $f_{0S}(\lambda) = u_{0S}\max(-\lambda,0) + l_{0S}\min(-\lambda,0)$. For $k = 1,\ldots,K$ and $S \in \Im_k$, define $\Lambda_{kS} = \mu_{kS} + \sum_{\{\hat{S}\in\Im_k | S\subseteq\hat{S}\}}\lambda_{k\hat{S}}$ and $f_{kS}(\lambda) = u_{kS}\max(\lambda,0) + l_{kS}\min(\lambda,0)$. For all $k = 0,\ldots,K$ and $S\in\Im_k$, f_{kS} is nonnegative and convex. Substituting into (4), we obtain

$$(5)\quad \pi(p) = \min_{\Lambda,\mu}\sum_{k=0}^{K}\sum_{S\in\Im_k-\{R_k\}} f_{kS}(\Lambda_{kS} - \Lambda_{kP_k(S)}) + \sum_{k=1}^{K} f_{kR_k}(\Lambda_{kR_k} - \mu_k) + f_{0R_0}(\Lambda_{0R_0})$$

subject to

$$\Lambda_{k_j\{j\}} - \rho_j\Lambda_{0P_0(j)} \geq v_j \text{ for all } j \in J$$

$$\mu_k \leq p_k \text{ for } k = 1,\ldots,K.$$

Notice that the dual constraints and objective simplify to this form because of the tree structure we have imposed. For $k_j = k$, the sets in tree $\Im_k$ that include j are exactly $\{j\}$, $P_k(\{j\})$, $P_k(P_k(\{j\})),\ldots,R_k$, and similarly for tree zero.

Because each f_{kS} is convex, each term of the objective in (5) is submodular in (Λ,μ,p) using the product order. The objective is a sum of submodular functions and therefore is itself submodular. A set $\{y | l \leq a\cdot y \leq u\}$ is a sublattice in the product order if and only if any two nonzero elements of the a-vector have opposite signs.[10] So, each constraint in problem (5) defines a sublattice on the set of possible (Λ,μ,p)-vectors, and the intersection of sublattices is a sublattice. Hence, by Lemma 2, $\pi(p)$ is submodular. And therefore, by Lemma 1, V is substitutable.

PROOF OF THEOREM 2:

Since the corresponding primal problem can be represented as a continuous, concave maximization on a compact set, the maximum exists and coincides with the minimum of the dual. Since the valuations are concave, the set of market-clearing prices is the set of solutions to the dual problem: arg $\min_p \sum_{n=1}^{N}\pi_n(p)$. Since each π_n is continuous and convex, the set of minimizers of the dual problem is closed and convex. Since each π_n is submodular, by a theorem of Donald M. Topkis (1978), the set of minimizers of the dual problem is a sublattice.

[10] This property of the *rows* of the *dual* constraint matrix, that no two nonzero entries have the same sign, is in remarkable correspondence with the condition required in the proof of Theorem 3 that no two nonzero entries in the *columns* of the constraint matrix of the *primal* problem have the same sign. The dual constraint matrix is obtained from the primal constraint matrix essentially by transposition, so the two conditions coincide. That is why the structure of assignment messages can be useful for proving the substitutes conclusion of Theorem 1 and the integer allocation conclusion of Theorem 3.

PROOF OF THEOREM 3:

To find $q^* \in \arg\max_{\{q \mid q_n \in Q_n\}} \sum_{n=1}^{N} \mathrm{V}_n(q_n)$ subject to $\sum_{n=1}^{N} q_{nk} = 0$, we substitute from (3) and introduce variables x_{nkS} as the sums of their successors in the tree (the elements of the set $P_{nk}^{-1}(S)$), so that the optimization is converted into one in which every inequality constraint involves just one variable. Because the bid j, and not just the set $S=\{j\}$, can be a successor to sets in the constraint trees under the extended predecessor function, define $x_{nkj} \equiv x_j$ for all $n = 1,\ldots,N$ and $k = 1,\ldots,K$. The tree structure allows us to show something stronger than claimed by the theorem, namely, that there is an integer optimal solution x^* to the resulting problem:

$$
\begin{aligned}
(6) \qquad & \max_q \sum_n V_n(q_n) && \text{subject to} \sum_n q_{nk} = 0 \quad \text{for } k = 1,\ldots,K \\
& = \max_x \sum_n \sum_{j \in J(n)} v_j x_j && \text{subject to} \\
& l_{nkS} \le \sum_{j \in S} x_j \le u_{nkS} && \text{for } S \in \Im_{nk},\, k = 1,\ldots,K, n = 1,\ldots,N \\
& l_{n0S} \le \sum_{j \in S} x_j \le u_{n0S} && \text{for } S \in \Im_{n0}, n = 1,\ldots,N \\
& \sum_n \sum_{j \in R_{nk}} x_j = 0 && \text{for } k = 1,\ldots,K \\
& = \max_x \sum_n \sum_{j \in J(n)} v_j x_j && \text{subject to} \\
& - x_{nkS} + \sum_{S' \in P_{nk}^{-1}(S)} x_{nkS'} = 0 && \text{for } S \in \Im_{nk}, k = 1,\ldots,K, n = 1,\ldots,N \\
& x_{n0S} - \sum_{S' \in P_{n0}^{-1}(S)} x_{n0S'} = 0 && \text{for } S \in \Im_{n0},\, n = 1,\ldots,N \\
& l_{nkS} \le x_{nkS} \le u_{nkS} && \text{for } S \in \Im_{nk},\, k = 0,\ldots,K,\, n = 1,\ldots,N \\
& \sum_n x_{nkR_{nk}} = 0 && \text{for } k = 1,\ldots,K.
\end{aligned}
$$

The sign restrictions $l_{nkS} \le 0$ and $u_{nkS} \ge 0$ ensure that $x \equiv 0$ satisfies the constraints of the problem, so the problem is feasible. The bounds on each variable imply that the constraint simplex is bounded. For a feasible, bounded linear program, there is always an optimal solution at a vertex of the constraint simplex.[11] Hence, to prove the theorem, it is sufficient to show that every vertex of the simplex defined by the constraints in (6) is an integer vector.

Each vertex of the constraint simplex is determined by a set of binding upper and lower bound constraints of the form $x_{nkS} = u_{nkS}$ or $x_{nkS} = l_{nkS}$ and the equation $Ax = 0$,

[11] See, for example, Gale (1960).

which describes the equality constraints in (6). Fix any vertex and denote the right-hand sides of the binding upper and lower bound constraints by $\overline{u}$ and $\overline{l}$, which, by hypothesis, are integer vectors. Write the vector x in the form $(\hat{x}, \overline{x}_l, \overline{x}_u)$, where the binding inequality constraints are $\overline{x}_l = \overline{l}$, $\overline{x}_u = \overline{u}$, which we write as $\overline{x} = (\overline{u}, \overline{l}) \equiv \overline{b}$. Let $\overline{A}$ and $\hat{A}$ be the matrices consisting of the columns of A corresponding to $\overline{x}$ and $\hat{x}$, respectively. Then the equation $Ax = 0$ can be written as $0 = Ax = \hat{A}\hat{x} + \overline{A}\,\overline{x} = \hat{A}\hat{x} + \overline{A}\,\overline{b}$. Taking $b \equiv -\overline{A}\,\overline{b}$, the equality constraints can be written as $\hat{A}\hat{x} = b$. Observe that b is an integer vector, because $\overline{A}$ is an integer matrix and $\overline{b}$ is an integer vector.

It is therefore sufficient to show that for every nonsingular submatrix $\hat{A}$ of A and every integer vector b, there is an integer solution $\hat{x}$ to $\hat{A}\hat{x} = b$. For this, it suffices to show that A is *totally unimodular.*[12] According to a theorem attributed to Alan J. Hoffman (see I. Heller and C. B. Tomkins (1956)), a matrix is totally unimodular if two conditions are satisfied: all the entries of A are elements of the set $\{0, +1, -1\}$, and any two nonzero entries in the same column have opposite signs. We finish by verifying these Hoffman conditions.

Examine the columns of A as represented in (6) which correspond to the variables x_{nkS}. For $k = 0$ and $S = R_{n0}$, the root of tree $\Im_{n0}$ for some participant n, x_{n0S} appears in only one equality constraint in (6), and so has the single entry $+1$ in its column. For $k = 1, \ldots, K$, each of the variables $x_{nkR_{nk}}$ appears twice (once in its defining equation and, again, in the market-clearing constraint for k), and its two coefficients, ± 1, have opposite signs. For $k = 1, \ldots, K$, and all sets $S \in \Im_{nk} - \{R_{nk}\}$, x_{nkS} appears twice: once with coefficient -1 in the equation defining x_{nkS} and once with coefficient $+1$ in the equation defining $x_{nkP_{nk}(S)}$. For $k = 0$ and $S \in \Im_{n0} - \{R_{n0}\}$, x_{n0S} appears twice: once with coefficient $+1$ in its defining equation and once with coefficient -1 in the equation defining $x_{n0P_{n0}(S)}$. Last are the x_j variables. Recall that by our extended definition of predecessor, $j \in P_{nk}^{-1}(S)$ for exactly two sets, one in $\Im_{nk_j}$ with coefficient $+1$ and one in $\Im_{n0}$, with coefficient -1. Hence, the Hoffman conditions are satisfied.

II. Partial Converse to Theorems 1 and 2

The structure of assignment messages allows bidders to report values and effectiveness coefficients without limitations but restricts the form of constraints to be a constraint forest. This section shows that if one weakens the restriction that $\Im_{n0}$ is a tree, then the conclusions of Theorems 1 and 2 fail.

The problem can be illustrated with an example of a buyer for whom the lower bounds l_j and l_{kS} are all zero. Suppose that there are three goods and that this buyer has three bids, $j = 1, 2, 3$, each with $v_j = 2.9$, $k_j = j$, and $u_j = 2$. Suppose that the multi-product group constraints in the problem are $x_1 + x_2 \leq 3$ and $x_2 + x_3 \leq 3$, violating the tree structure. Then, for the price vector $(0, 1, 2)$, the corresponding demand is $(2, 1, 2)$ and for the price vector $(3, 1, 2)$, the corresponding demand is $(0, 2, 1)$; raising the price of good 1 reduces the demand for good 3, violating the substitutes condition. Moreover, if the available quantities are one unit of good 2 and two units each of goods 1 and 3, then the market clears for price vectors $(0, 1, 2)$ or

[12] See the *Wikipedia* entry (http://en.wikipedia.org/unimodular_matrix) on "unimodular matrix" for an accessible treatment of the relevant mathematics.

$(2,1,0)$ but not for the join, which is $(2,1,2)$, so the set of market-clearing prices in this example is not a sublattice.

More generally, given *any* set of constraints $\Im_{n0}$ that fails to have the tree structure, we can find a similar counter example as follows. Since the constraints do not form a tree, there are two sets, $S, S' \in \Im_{n0}$, such that each of the three disjoint sets $S - S'$, $S \cap S'$, and $S' - S$ are nonempty. Let goods 1, 2, and 3 denote elements of these three sets and specify that the values of any other goods are zero. Let the bounds constraining these goods be given as in the preceding paragraph and let the bounds on all other constraints be very large, so that those constraints do not bind. This specification reproduces the example of the preceding paragraph starting from any $\Im_{n0}$ that is not a tree. That proves the following theorem.

THEOREM 4: *If the set $\Im_{n0}$ is not a tree, then there exist bids and integer bounds for each $S \in \Im_{n0}$ and supplies for the other participants, such that the valuation V_n is not a substitutes valuation, the indirect profit function π_n is not submodular, and the set of market-clearing prices is not a sublattice.*

III. Tightness

A *direct mechanism* is a triple (N, M, ω), where N is the set of participants, M is the product space of types ("message profiles"), and $\omega: M \rightarrow \Omega$, where Ω is the set of possible outcomes. The mechanism $(N, \hat{M}, \omega)$ is a simplification of the mechanism (N, M, ω) provided $\hat{M} \subseteq M$. For tightness analysis, it is assumed that $\Omega \subseteq \times_{n \in N} \Omega_n$ where each Ω_n is a topological space, and that each player n's payoff is represented by a continuous function $u_n : \Omega_n \rightarrow \Re$.

A simplified direct mechanism has the *outcome closure property* if, for every player n, strategy profile $\hat{m}_{-n} \in \hat{M}_{-n}$, strategy $m_n \in M_n$, and every open set $O \subset \Omega_n$ such that $\omega_n(m_n, \hat{m}_{-n}) \in O$, there is a strategy $\hat{m}_n \in \hat{M}_n$, such that $\omega_n(\hat{m}) \in O$. This means that when other participants are limited to using simplified messages, limiting n to do the same has little or no effect on the set of outcomes that n can produce. The mechanism $(N, \hat{M}, \omega)$ is a tight simplification of (N, M, ω) if for all utility profiles $u = (u_n)_{n \in N}$ and every $\varepsilon \geq 0$, every pure-strategy profile that is an ε-Nash equilibrium of the simplified mechanism is also an ε-Nash equilibrium of the original, extended mechanism. The Simplification Theorem of Milgrom (forthcoming) asserts that if $(N, \hat{M}, \omega)$ has the outcome closure property with respect to (N, M, ω), then the simplification is tight.

For this application, we take $\omega_n = (q_n, p)$. This specification permits each participant to care about his own goods assignment and the prices, but not about the goods assigned to others. In standard equilibrium theory, preferences for a participant n depend only on $(q_n, p \cdot q_n)$, his goods assignment, and payment. By including the price vector in a more general way, the tightness analysis allows that a participant may prefer that its competitor's product commands a low price or that its partner's product commands a high price. It also allows a participant to have any preference for which the preferred sets are all closed and convex, but participants are not limited to such preferences and certainly not just to the preferences that are describable using assignment messages.

The next theorem applies not just to the full assignment exchange, but also to mechanisms that limit the messages participants can use to a subset of the assignment messages. To describe the permissible limitations on messages, let us say that an assignment message m_n is *minimally constrained* if its only finite constraint bounds (l_{kS}, u_{kS}) correspond to the singleton sets $S = \{j\}$. An *elementary assignment message* m_n for participant n is an assignment message that is minimally constrained and includes, at most, two bids for any product k: $|\{j \in J(n) : k_j = k\}| \leq 2$ for $k = 1, \ldots, K$. A *full Walrasian exchange* is any mechanism that accepts messages describing, for each participant, closed convex preferences over net trades and a feasible consumption set with the null trade in its interior; and maps any message profile into a corresponding competitive equilibrium outcome, whenever one exists.

THEOREM 5: *Any simplified Walrasian exchange in which each bidder n's message space contains only assignment messages, and contains all elementary assignment messages, satisfies the outcome closure property with respect to any full Walrasian exchange and* (*hence*) *is a tight simplification.*

Theorem 5 is proved by showing that for any price vector and goods assignment that can be obtained by some general message, a buyer can acquire nearly the same bundle and bring about nearly the same prices with an elementary assignment message that bids for the equilibrium quantities at slightly higher than the equilibrium prices and that bids for additional quantities at slightly lower prices. For the full proof, details are added to ensure that this construction applies not only to buyers but also to sellers and to participants who bid to buy some items and to sell others. This establishes the outcome closure property. The tightness conclusion then follows from the Simplification Theorem.

PROOF:

Let $\hat{M}_n$ be bidder n's simplified message space, and let M_n be the message space used by a full Walrasian mechanism, as described above. Fix a participant n and messages $\hat{m}_{-n} \in \hat{M}_{-n}$ and $m_n \in M_n$. Let $(p, q) \equiv \omega(\hat{m}_{-n}, m_n)$. We now construct the elementary message described informally in the preceding paragraph.

Let $\sigma_{nk} = sign(q_{nk}) \in \{-1, 0, 1\}$ and fix $\varepsilon > 0$. Since n's message space includes all elementary assignment messages, it includes the message $\hat{m}_n$ with bids $j = 1, \ldots, 2K$ as follows. For $j = 1, \ldots, 2K$, let $k_j = \lceil j/2 \rceil$ (the smallest integer weakly exceeding $j/2$) and set $v_{2k-1} = p_k + \sigma_{nk}\,\varepsilon$, $v_{2k} = p_k - \sigma_{nk}\varepsilon$, $u_{2k-1} = u_{2k} = \max(0, q_{nk})$ and $l_{2k-1} = l_{2k} = \min(0, q_{nk})$. The message $\hat{m}_n$ specifies no other finite bounds. Let $(\hat{p}, \hat{q})$ be the competitive equilibrium outcome selected by the full Walrasian mechanism when the message profile is $\hat{m}$.

Since (p, q) is a competitive equilibrium for the report profile $(\hat{m}_{-n}, m_n)$, $q_n \in \arg\max_{y_n} \left[\max_{\{y_{-n} | \sum_{l \neq n} y_l = -y_n\}} \left(V_n(y_n | m_n) + \sum_{l \neq n} V_l(y_l | \hat{m}_l)\right)\right]$. And since n demands q_n at prices p, (p, q) is also a competitive equilibrium for report profile $\hat{m}$. From that and the fact that $\varepsilon > 0$, q_n *uniquely* solves $\arg\max_{y_n} \left[\max_{\{y_{-n} | \sum_{l \neq n} y_l = -y_n\}} \left(V_n(y_n | \hat{m}_n) + \sum_{l \neq n} V_l(y_l | \hat{m}_l)\right)\right]$. Hence, even though there may be multiple competitive equilibria for the message profile $\hat{m}$, all assign the bundle q_n to participant n: $\hat{q}_n = q_n$. Moreover, since every market-clearing price vector supports this choice by n, the price vector $\hat{p}$ must satisfy $p_k - \varepsilon \leq \hat{p}_k \leq p_k + \varepsilon$ for every product k. Since ε can be arbitrarily small, the outcome closure property is proved. Tightness then follows from the Simplification Theorem cited above.

IV. Connections to Two Familiar Mechanisms

In case $K = 1$, each participant's assignment message describes a step supply or demand function. The assignment exchange is then a familiar double auction, in which the allocation is determined by intersecting single-product supply and demand curves. When the market-clearing prices or quantities are not unique, any selection rule is consistent with the assignment exchange.[13] In general, the assignment exchange extends the single-product double auction by allowing multiple products and a rich set of substitution possibilities among them.

The integer assignment exchange is connected to the Vickrey auction. In a Vickrey auction, if a participant n acquires a single unit of a single good k, its payment is the opportunity cost of that good, which is equal to the incremental value of one additional unit of good k to the coalition of all *other* participants. In the linear program for the integer assignment exchange, the lowest market-clearing price p_k for good k is its lowest dual price—the amount by which the optimal value would increase if an additional unit of good k were made available to the coalition of *all* players. If participant n has demand for just one unit in total and acquires a unit of good k, then the additional unit for the coalition of all participants is actually assigned to someone besides n, so p_k is the increased optimal value of that unit to the other participants—n's Vickrey price.

THEOREM 6: *Suppose that some participant n bids to acquire, at most, one unit in an integer assignment exchange, and that the exchange selects the price vector p that is the minimum market-clearing price vector. Then, if n acquires a unit of good k, the price p_k is equal to n's Vickrey payment.*

A symmetric statement can be made about participants who *sell* one unit and exchanges that select the *maximum* market-clearing price vector.

V. Likely Applications

The most immediate opportunity for application of the assignment exchange technology is to auction off two or more substitute products for which the length-of-report problem is important. Paul D. Klemperer (2008) has independently proposed a simple version of the assignment auction design. For this section, an *auction* is simply an exchange with one seller and many buyers or one buyer and many sellers.[14]

[13] In one-sided cases (with just bids to buy and a fixed supply, or bids to sell and a fixed demand), the kinds of problems found in share auctions (Robert B. Wilson (1979)) can present themselves. Typical solutions to these problems, such as those proposed in David McAdams (2002) and Ilan Kremer and Kjell G. Nyborg (2004), can be adapted to the assignment exchange.

[14] Assignment auctions have several variations, mirroring the variations common in other sealed-bid auctions. For example, the auctioneer (whether buyer or seller) may move first, possibly announcing target quantities or reserve prices, or a supply or demand curve, or perhaps announcing rates of substitution among products. Or, there may be multiple stages, for example, a qualifying stage with just some bidders invited to the second stage.

The previous best-practice mechanisms for dealing with the length-of-report problem were sequential mechanisms—the simultaneous ascending and descending clock auctions (Ausubel (2001)). In simultaneous clock mechanisms, bidders are asked to report supplies or demands at each of a sequence of announced prices, and the reported information is used to find approximate market-clearing prices and allocations. Because demands are announced for only a finite number of price vectors, the information reported is much less than that of a full direct mechanism.

Simultaneous ascending or descending multi-product auctions of various kinds have been used for several high-value applications, most commonly ones involving radio spectrum, electricity, or natural gas (Milgrom (2004)), but also for real estate transactions and certain agricultural commodities markets.[15] When the goods for sale are substitutes and participants bid myopically, various versions of the simultaneous ascending or descending auctions have been found not only to economize on communications but also to identify allocations that are efficient or stable or to find minimum or maximum market-clearing prices (Alexander S. Kelso, Jr. and Crawford (1982); Gul and Stacchetti (2000); Milgrom (2000); Ausubel (2004), Milgrom and Strulovici (2009)). This property makes these auctions directly comparable to assignment auctions.

Because simultaneous ascending and descending auctions economize on communications and enable bidders to substitute in response to changing prices, they have important advantages over independent auctions of different goods. But they also have properties that make them unsuitable for many applications. Four of these disadvantageous properties are high participant costs, long times-to-completion, imprecise computations, and difficulties of scheduling. Any multi-round, real-time process adds the cost of real-time bidding to the costs of preparing for the auction. In current practice, dynamic auctions for gas and electricity take several hours to reach completion, while spectrum auctions take days, weeks, or even months. Such long times-to-completion cripple these mechanisms for the most time-sensitive markets, such as hour-ahead power markets, where only minutes are available to complete an exchange. In practice, ascending and descending auctions fail to identify exact market-clearing prices because they change the direction of price increments only a small number of times, using discrete price increments.[16] Finally, in export markets, where potential buyers may reside in a dozen or more different time zones, scheduling a convenient time for several hours of real-time bidding may be impossible. These four problems are avoided by direct mechanisms, including simplified direct mechanisms like the assignment auction.

The two main practical limitations of assignment exchanges arise because the message space may be too narrow to express bidders' actual preferences and because, as a static mechanism, the auction provides no opportunity for bidders to learn from competing bids. The latter can be significant when there is uncertainty about a common factor that raises or lowers all values together, or when a bidder's preferred trades depend on the trades made by other bidders.

[15] In a simultaneous ascending auction, prices can be called by the auctioneer (these are the so-called "clock auctions") or by individual bidders.

[16] Ausubel and Cramton (2004) show how a clock auction with a richer message space ("intra-round bidding") can avoid some of the disadvantages of discrete price increments.

Even the integer assignment messages, with their limited one-for-one substitution, allow ample expressiveness for some applications. Suppose, for example, that an electricity buyer can purchase power from any of three sources, $k \in \{1, 2, 3\}$, subject to transmission costs (t_1, t_2, t_3) and transmission capacity limits (U_1, U_2, U_3). If the buyer needs to buy P units of power and the value per unit is α, then bids $j = 1, 2, 3$ with $k_j = j$, $v_j = \alpha - t_j$, $u_j = U_j$, $l_j = 0$, and one constraint for $S = \{1, 2, 3\}$ with $u_{0S} = P$ and $l_{0S} = 0$ accurately express the bidder's demand. If there are also significant transmission losses from some source j, a general assignment message accommodates those by allowing the bidder to set $\rho_j < 1$.

In a double-auction with multiple buyers and sellers of electric power, other kinds of assignment messages can be valuable. For even if a buyer has already filled all of its power needs for some time period, it may be willing to sell up to β units of power at source 1 and buy the same quantity at 2 or 3, provided the price differential is favorable. This swap can be encoded with three bids and the constraints: $0 \geq x_1 \geq -\beta$, $\beta \geq x_2, x_3 \geq 0$, $x_1 + x_2 + x_3 = 0$.

Swap bids have the potential to add liquidity to an exchange hindered by lack of volume. Investigating this fully is beyond the scope of this paper. It requires a theory of why owners do not constantly participate in, and provide liquidity to, markets. Nevertheless, it is clear that in a market with modest liquidity, swaps encourage participation by limiting the risk that one part of an intended transaction might be executed without the other parts. With separate markets, a swapper with a budget limit might have to sell one commodity before buying the other in order to raise funds to transact, leaving the swapper exposed to the risk of not finding a seller for the other part of the planned transaction. By eliminating such risks, swaps make participation safer, increasing liquidity.

The power of simple assignment messages in the preceding example is important because simplicity is often a design goal. One might simplify the general assignment exchange by limiting the number of bids, constraints, or levels in the constraint trees. Theorems 1, 2, 3, and 5 have been constructed to apply even to exchanges that incorporate such additional simplifications.

One common limitation imposed by auctioneers is a credit limit on buyers. Buyers might also want to express a budget limit. The assignment message space does not allow this to be done directly, but it does allow surrogates, such as a limit on the maximum total bid from a bidder, or on the maximum quantities that can be demanded.

Maximum quantity limits on some bidder or set of bidders can also be useful for a government auctioneer when bidder market power is a concern, or when there is a goal of promoting entry. Sometimes, this goal is best implemented by careful product definitions. For example, if the auctioneer wants to limit bidders 1 and 2 to purchase no more than half of the available units of good 1, it can accomplish that by splitting good 1 into types 1A and 1B and restricting bidders 1 and 2 from bidding on type 1B. This procedure is similar to the set-asides used by the FCC to restrict purchases by incumbents in some radio spectrum auctions.[17]

[17] The FCC combined this with restrictions on post-auction transfers to limit gaming of the system.

Whether the assignment messages are sufficiently encompassing is likely to vary by application. Certainly, scale economies and complements among lots are sometimes important and cannot generally be solved by redefining lots. For example, in electricity, generating plants typically have large fixed costs that require all or nothing decisions about whether to use their power capacity. While such limits are not directly expressible using assignment messages, it is often possible to use the assignment exchange as part of a solution. One ad hoc procedure is to operate the exchange in two or more rounds to allow preliminary price discovery to guide bids at the final round. This does not entirely eliminate the fixed-cost problem, but it may sometimes mitigate it sufficiently. Staged dynamics of this sort may also be helpful when there are important common value elements or when bidders can invest in information gathering during the process, as in Olivier Compte and Philippe Jehiel (2000) or Leonardo Rezende (2005).

Three key properties of assignment and integer assignment messages—that they are simple to use, express only substitutable preferences, and that integer assignment messages lead to efficient integer solutions—make them potentially valuable for use with other mechanisms in addition to the Walrasian exchange. For example, two principal disadvantages of "standard" Vickrey auctions—the length-of-report problem and "low" seller revenues (less than in any core allocation)—hinge on the requirement to report a separate value for each possible package and the availability of messages that report nonsubstitutable values, respectively.[18] A simplified Vickrey auction in which bidders are limited to reporting assignment messages escapes both of these disadvantages. There may also be applications to matching problems, without cash transfers, such as the problems of assigning students to courses or flight attendants to routes, where integer allocations are essential and the substitutes structure may be a reasonable approximation.[19]

Simplification represents a promising approach to applied mechanism design, and assignment messages show high potential for use in simplified mechanisms for trading substitutable goods. Exchanges that utilize assignment messages are tight, easy for bidders to use, quick to run, precise in determining both equilibrium prices and goods assignments, and adaptable to settings that require integer solutions. The assignment exchange design is *robust*, in the sense that its key properties remain intact even when the assignment message space is further restricted in any way that does not eliminate any *elementary* assignment messages. It is also *maximal* in the sense that no extension of the constraint tree architecture is possible without destroying the key substitutes property of the message space. Taken together, these attributes make the assignment exchange an attractive candidate for the many practical applications.

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[18] Milgrom (2004), sections 2.5 and 8.1 and Ausubel and Milgrom (2006).

[19] Eric Budish et al. (2008) have begun to study this problem.

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